From models to infinity

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The references are this one stack exchange post and the books [DS95] and [Rie].

The goal is to understand in what sense we can "replace" model categories with ∞-categories. The first step is to understand the following quote from the nlab page on qullen equiviliences:

> Quillen equivalent model categories present equivalent (∞,1)-categories.

we also sustained long debates over the motivation for the definition of Quillen equivilience and so we might say something as to what that all came to.

1 Quillen Equivalence

The definition of Quillen equivalence (QE) upon first look appears *ad hoc* to me. The naive guesses I would have had were something of the form:

- (Strongest / Everything) An equivalence of categories, where the functors preserve the model structure.
- (Weak homotopy) A functor that preserves the model structure and induces a homotopy equivalence.
- (Homotopy) Formulate something subtle using strucutre preserving functors between the model category crossed with the one simplex as a category?

Upon reflection I realise why; given a structure in mathematics we *usually* say that the right notion of morphism is one which preserves *all* the structures. Examples are plentiful and the reason is straight forward if we didnt want the maps to preserve the structure we would simply drop the structure from the definition, make it an extra property or something else etc.

This is not so for (QE), we dont require the functor to preserve all of the model structure, the left adjoint preserves only cofibrations and the right adjoint only fibrations. The thing to gripe with is really that there is *no notion* of a category of model categories; the morphisms are either left or right morphisms, there are *two notions of category*, and an equivalence is forcing them to interact. Recall that an adjoint pair is exactly half of an equivilence of categories, an equivilence is four natural transformations and an adjoint is two. In normal settings we also require a pair of maps

$$f : G \to H : h$$

say a smooth map with a smooth inverse before we say two objects are isomorphic, this is because the groups or spaces or what have you G, H have no concept of left or right. For model categories the notion of being on the left or right is *part* of the data. In particular [DS95, Prop 3.13] tells us that cofibrations are characterised by their left lifting properties (with respect to acyclic fibrations), and fibrations are characterised by their right lifting property (of acyclic cofibrations). Thus a Quillen adjunction is a pair of maps such that the *left function* preserves *left lifting properties* and the right map preserves right lifting properties.

In particular **being Quillen equivalent is not an equivalence relation**, in the sense that if there is an equivalence $X \to Y$ there is not necessarily one $Y \to X$. The actual relation generated will be an equivalence one because we will just say that some pair exists and will be agnostic about which side X or Y are on of the maps. This is an odd notion of equivalence indeed, one that depends on left and right, that is not symmetric. This fact that precludes symmetry means that no other equivalence is like this, however there are no other examples of things which have a natural left and right structure, thus necessitating one on their morphisms.

In fact there is one other example of such a relation, namely the notion of weak homotopy equivalence. If we wanted to insist on a parallel to weak homotopy equivalence we would fail because there the idea of map between spaces is not of this form it is that of continuity and has not handedness to it (left or right), however this is probably still worth noting. What I dont think is happening, but what should be happening is a definition that ascends in the sense that we should be able to put a model structure on the category of model categories (fibrant = right quillen functor?) and then the notion of fibration and weak equivilence generates its own concept at the level of the higher category.

Example. *// Of the failure of symmetry*

Other things that are worth mentioning in this regard are the following thing that people mentioned to me but that I havent fleshed out yet:

- The following quote from [DS95] introduction:

  > "Quillen called the study of model categories homotopical algebra and thought of it as a generalization of homological algebra"

  This gives us the clue about what is really going on here. Quillen is trying to preserve the algebra of homotopy not just the homotopy categories etc. but precisely how you do the algebra, its constructions and details.

- Requiring a full equivalence would mean that there are very few such equivalences but also remove the importance from the homotopy category.

- Apparently the ability to derive functors by doing fibrant and cofibrant replacements is the thing that is really being preserved here, so the structure we preserve is that of derived functors somehow.

After typing all this it appears to me that the real question might be why dont we require Quillen functors to preserve both fibrations and co-fibrations.

Given the content of these notes then I will prefer to think of this as merely a sufficient condition for two infinity categories to be equivalent (which I believe is an equivalence relation). Thus even though I dont have a good answer to this question in some sense ∞-categories have saved me from the need of giving one; I will soon forget model categories were anything but a tool.

2 The $(\infty, 1)$-category presented by a model category

There are apparently several equivalent ways to describe this construction. What im copying is this one stack exchange post. It makes many claims and references only the following texts [BK11], [Toe05] and Luries two books. The paper [DHK97] We will try to write down what they claim and track down the references and see if this is satisfactory. For the time being we will mark a claim that needs to be justified with ⚠️.

Throughout we let (M, F, C, W) be a model category with its set of fibrations, cofibration and weak equivalences, and we denote (M, W) the category with only its weak equivalences. This is an example of a so called *relative category*, that is a category with a class of weak equivilences.

2.1 Hammock Localization

I believe details can be found in [DHKS05]. The Hammock localization is a functor

$$L^H : \mathrm{RelCat} \to \mathrm{Cat}_\Delta$$

sending a category with weak equivilences to a simplicially enriched category.⚠️Define, properties of faithfullness. Then we can apply the coherent nerve to get the required infinity category. ⚠️Locally Kan? Also the poster takes the derived functor of Lh but the domain is not the homotopy category, might be a conflict of notation idk?

Remark: Not all infinity categories are presented by model categories. ⚠️Reference and example please.

2.2 Complete Segal Spaces

Another way to realize a simplical set from a relative category is to create a bi-simplical set, or a complete segal space and then use the standard Quillen equivalence to produce a simplical set. This is done for instance in this section of the nlab. In fact with appropriate model structures this is itself a Quillen equivalence between relative categories and complete Segal spaces.

References

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[DS95] W.G. Dwyer and J. Spalinski. Homotopy Theories and Model Categories. In *Handbook of Algebraic Topology*, pages 73–126. Elsevier, 1995.

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